REMARKS ON THE SINGULARITY OF ELASTIC SOLUTIONS NEAR CORNERS

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A. I. KALANDIIA
(Tbilisi)

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It is known that in the plane case elastic displacements and stresses can be expressed through the stress function of Airy $U(x, y)$ and one harmonic function $p(x, y)$ in the following form:

$$\sigma_x = \frac{\partial^2 U}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 U}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y} \tag{1.1}$$

$$2\mu u = -\frac{\partial U}{\partial x} + \frac{2(\lambda + 2\mu)}{\lambda + \mu} p, \qquad 2\mu v = -\frac{\partial U}{\partial y} + \frac{2(\lambda + 2\mu)}{\lambda + \mu} q \tag{1.2}$$

where the functions U and p are connected with each other through relationship

$$\Delta U = 4 \frac{\partial p}{\partial x} \tag{1.3}$$

while q denotes a harmonic function which is conjugate with p.

Let us introduce the plane of the complex variable $z = x + iy = re^{i\vartheta}$.

Williams [1] studied the stresses near the vertex of the sector with linear sides $\vartheta = 0$ and $\vartheta = \alpha$, where α is an arbitrary angle; $0 < \alpha \leqslant 2\pi$. In the investigation of Williams the starting equations were particular solutions of the theory of elasticity in the form

$$U = r^{\lambda+1}[b_1 \sin(\lambda + 1)\vartheta + b_2 \cos(\lambda + 1)\vartheta + b_3 \sin(\lambda - 1)\vartheta + b_4 \cos(\lambda - 1)\vartheta] \tag{1.4}$$

$$p = r^m [a_1 \cos m\vartheta + a_2 \sin m\vartheta] \tag{1.5}$$

where λ and b_i are arbitrary complex constants; m, a_1 and a_2 are also constants which are determined as indicated above from (1.3).

Putting stresses and displacements together according to above equations on radial sides of the segment, the above mentioned author satisfied homogeneous conditions of the plane problem and obtained linear equations for the determination of unknown b_i. Having then constructed the characteristic equation for λ and having determined its proper root λ, he found the nontrivial field of elastic elements which for some values of angle α gave unbounded stresses at the vertex. By this method for a given material the dependence was established by Williams between the order of singularity of stresses near the corner and the magnitude of its angle in the case when on the sides of the sector which form the angle, external forces or displacements are given, or in the case when the stresses are given on one side and the displacements on the other (basic mixed problem).

A somewhat different approach to the construction of characteristic equations is indicated below, and the studies of Williams are extended by the examination of other possible forms of boundary conditions (*).

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*) It should be noted that at the Third All-Union Mechanics Congress (Moscow, January and February 1968) a general result on construction of solutions for plane problems near corners was presented in the communication of I. I. Vorovich.

2. First of all let us represent Eqs. (1.4) and (1.5) in the form

$$2U = c_1\bar{z}z^{\lambda} + c_2z\bar{z}^{\lambda} + c_3z^{\lambda+1} + c_4\bar{z}^{\lambda+1} \tag{2.1}$$

$$2p = c_1z^{\lambda} + c_2\bar{z}^{\lambda} \qquad (2iq = c_1z^{\lambda} - c_2\bar{z}^{\lambda}) \tag{2.2}$$

where c_i are arbitrary complex constants.

Necessary and sufficient conditions for the right sides of previous equations to be real apparently will be

$$c_1 = \bar{c}_2, \qquad c_3 = \bar{c}_4, \qquad \mathrm{Im}\ \lambda_i = 0 \tag{2.3}$$

Only for these conditions do Eqs. (1.1) and (1.2) determine the real field of displacements and stresses.

The characteristic equations of Williams can therefore be constructed starting with boundary conditions of problems in the complex form, if

$$\varphi(z) = az^{\lambda}, \qquad \psi(z) = bz^{\lambda} \tag{2.4}$$

are taken as the Kolosov-Muskhelishvili potentials in the case of complex a, b and λ, if λ is formally considered a real quantity, and if the constants a, $\bar{a}$; b and $\bar{b}$ are considered as mutually independent unknowns.

For example, in the case of the first problem when the boundary condition has the form [2]

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t) \tag{2.5}$$

we shall have on the straight edges of the sector (*)

$$az^{\lambda} + \bar{a}\lambda z\bar{z}^{\lambda-1} + \bar{b}\bar{z}^{\lambda} = 0 \qquad \text{for} \quad \vartheta = 0,\ \alpha$$

By adding to the previous equations two others which are obtained through transition to conjugate values in the original equations, we shall have a system of four homogeneous equations. The requirement of the presence of a nontrivial solution for $a, \bar{a}, b$ and $\bar{b}$ in the system results in the following characteristic equation with respect to λ

$$\sin\lambda\alpha = \pm\lambda\sin\alpha \tag{2.6}$$

which coincides with the equation in [1] for this case.

Having determined, according to Williams the (generally speaking complex) root λ of Eq. (2.6) with the minimum positive real part and having found the corresponding unknowns a, b, $\bar{a}$ and $\bar{b}$, it is possible to construct a solution of the plane problem

$$2U = a\bar{z}z^{\lambda} + \bar{a}z\bar{z}^{\lambda} + \frac{b}{\lambda+1}z^{\lambda+1} + \frac{\bar{b}}{\lambda+1}\bar{z}^{\lambda+1}$$

$$2p = az^{\lambda} + \bar{a}\bar{z}^{\lambda} \qquad (2iq = az^{\lambda} - \bar{a}\bar{z}^{\lambda})$$

*) Conditions on the arc part of the boundary which have no effect on the character of stresses near the vertex of the sector will not be taken into consideration

This solution satisfies all requirements of classical elasticity. Stresses corresponding to this solution for $\operatorname{Re}\lambda < 1$ will have a singularity of the order of $1 - \operatorname{Re}\lambda$ near the vertex.

Together with basic problems of elastic theory we shall examine the case when on the radial sides of the sector the condition of contact without friction with an elastic section of a given shape exists. This condition, as is well known, consists of prescribing the normal displacement and the contact stress

$$v_\theta = f(t), \qquad \tau_{r\theta} = 0 \qquad (\theta = \text{const}) \tag{2.7}$$

The boundary conditions of the plane problem for given external stresses will be denoted by the symbol I. Boundary conditions of the problem with given displacements will be denoted by symbol II. The conditions for pressure of rigid punch without friction will be indicated by symbol III. The case where on one side of the sector condition I is given and on the other condition II, will be denoted by I-II etc.

Let us write known equations of complex representation in polar coordinates

$$2\mu v_\theta = \operatorname{Im}\{e^{-i\theta}[\varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}]\} \tag{2.8}$$

$$\tau_{r\theta} = \operatorname{Im}\{e^{2i\theta}[\bar{z}\varphi''(z) + \psi'(z)]\} \tag{2.9}$$

Differentiating (2.8) with respect to r along the ray $\theta = \text{const}$ and combining it with (2.9) we find

$$2\mu\frac{dv_\theta}{dr} - \tau_{r\theta} = \operatorname{Im}\{\varkappa\varphi'(z) - \overline{\varphi'(z)}\}$$

Consequently, in the case under examination boundary conditions (2.7) of the rigid punch problem can be represented in the form

$$\operatorname{Im}\varphi'(t) = \frac{2\mu}{\varkappa + 1}e^{i\theta}f'(t), \qquad \operatorname{Im}\{e^{-i\theta}[\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}]\} = 2\mu f(t) \tag{2.10}$$

$$\theta = 0, \quad \alpha \ (t = re^{i\theta})$$

Proceeding with homogeneous conditions (2.10) and potentials (2.4) we find the following characteristic equations for the case of contact with the punch along both sides, in completely the same manner as in the derivation of (2.6):

$$\sin\lambda\alpha = \pm\sin\alpha \tag{2.11}$$

Characteristic equations for the remaining cases I-III and II-III can be constructed in the same manner if on one radial side the condition (2.10) is given, and on the other the condition (2.5), or condition II

$$\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} = 0$$

These equations have the form

$$\sin 2\lambda\alpha = -\lambda\sin 2\alpha \quad \text{(for the case I-III)} \tag{2.12}$$

$$\sin 2\lambda\alpha = \frac{\lambda}{\varkappa}\sin 2\alpha \quad \text{(for the case II-III)} \qquad \left(\varkappa = \frac{3-\nu}{1+\nu}\right) \tag{2.13}$$

Here ν is the Poisson's ratio.

Roots of Eqs. (2.11) to (2.13) were found with minimum positive real parts for various values of angle α and $\nu = 0.3$. Plots of the function min $\operatorname{Re}\lambda$ are shown in

Fig. 1 for all six cases (*).

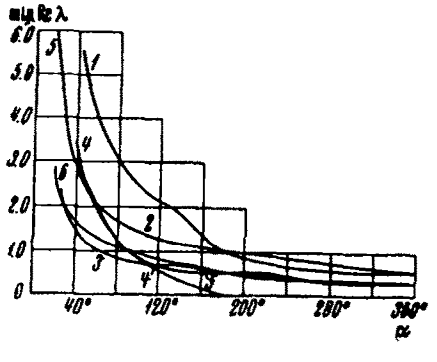

Fig. 1

In the case III-III (Eq. 2.11) the graph is shown only in the interval $(0, \pi)$. The root $\lambda = 0$, corresponding to $\alpha = \pi$, is trivial and should be excluded.

In the case III-III and also in cases when the vertex of the angle is the point of change in boundary conditions the singularity in stresses may occur at any angle α, greater than the right angle ($\alpha > {}^1/_2\pi$, $\alpha \neq \pi$). The case I-II presents an exception (Williams [1]) where the singularity arises already at $\alpha > 63°$.

The maximum order of singularity in stresses is the same for all three mixed cases. It is achieved for $\alpha = 2\pi$ (tip of section) and is equal to ${}^3/_4$ (Re $\lambda = {}^1/_4$).

3. In the case when in the construction of basic equations of elasticity, couple stresses (moments per unit area) are taken into account according to the concept of the brothers Cosserat [3], it is not difficult, for a given value of angle α, which is different from π and which exceeds some α_0, to construct the solution for the correctly formulated plane problem with unbounded stresses near the corners.

Let us examine for example the problem of contact of an elastic body with a rigid section of given shape in the absence of friction forces (problem III) in the previous section). As the third condition on the boundary let us impose the condition of giving the rotation of its points.

Let us write in polar coordinates the equations for the general representation of elements of elastic fields in the unsymmetrical case [4] :

$$\sigma_r + \sigma_\theta = 2\,[\varphi'(z) + \overline{\varphi'(z)}]$$

$$\sigma_\theta - \sigma_r + i(\tau_{r\theta} + \tau_{\theta r}) = 2\left[\bar{z}\varphi''(z) + \psi'(z) + m\varphi'''(z) + 2i\frac{\partial^2 H}{\partial z^2}\right]e^{2i\theta}$$

$$\tau_{r\theta} - \tau_{\theta r} = -\frac{1}{l^2}H$$

$$2\mu(v_r + iv_\theta) = \left[\varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - m\overline{\varphi''(z)} + 2i\frac{\partial H}{\partial \bar{z}}\right]e^{-i\theta}$$

$$M_\theta + iM_r = \left[m\varphi''(z) + 2i\frac{\partial H}{\partial z}\right]e^{i\theta}, \qquad 2\mu\omega = \mathrm{Im}\left\{(\varkappa + 1)\varphi'(z) + \frac{i}{2l^2}H\right\} \tag{3.1}$$

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \qquad m = 8(1-\nu)l^2 \tag{3.2}$$

Here M_r and M_θ are polar components of couple stress, ω is the rotation of an element of the medium.

The function H satisfies the Helmholtz equation in the region occupied by the elastic medium

$$\Delta H - k^2 H = 0, \qquad k^2 = \frac{1}{l^2} \tag{3.3}$$

where l^2 is a new constant of the material equal to the ratio of the modulus of bending to the modulus of displacement μ. The remaining notations are the same as before

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*) Curves for cases I-I, II-II and I-II were taken from the above cited work of Williams. The following numbering system was adopted for the curves: 1 (I-I), 2 (II-II), 3 (I-II), 4 (III-III), 5 (I-III) and 6 (II-III).

or generally known. From (3.1) we have

$$
\begin{aligned}
\tau_{\vartheta r} &= \operatorname{Im}\left\{e^{2i\vartheta}\left[\bar{z}\varphi''(z)+\psi'(z)+m\varphi'''(z)+2i\frac{\partial^2 H}{\partial z^2}\right]+\frac{ik^2}{2}H\right\} \\
-2\mu v_\vartheta &= \operatorname{Im}\left\{e^{i\vartheta}\left[\varkappa\overline{\varphi(z)}-\bar{z}\varphi'(z)-\psi(z)-m\varphi''(z)-2i\frac{\partial H}{\partial z}\right]\right\}
\end{aligned}
\qquad (3.4)
$$

Differentiating the second equation along the straight line $\vartheta = \mathrm{const}$ and combining it with the first equation we find

$$
2\mu\frac{dv_\vartheta}{dr}-\tau_{\vartheta r}=\operatorname{Im}\{(\varkappa+1)\,\varphi'(z)\} \qquad (3.5)
$$

Consequently, for the problem being examined, the homogenous boundary conditions prescribed along the radial sides of the sector, i.e. conditions

$$
v_\vartheta=\tau_{\vartheta r}=\omega=0
$$

can be represented in the form

$$
\begin{aligned}
&\operatorname{Im}\{e^{i\vartheta}[\varkappa\overline{\varphi(t)}-\bar{t}\varphi'(t)-\psi(t)-m\varphi''(t)]\}-\frac{dH}{dr}=0 \\
&\operatorname{Im}\varphi'(t)=0, \qquad H=0 \qquad \text{for } \vartheta=0,\alpha
\end{aligned}
\qquad (3.6)
$$

It is easy to convince oneself that the above conditions are satisfied by functions

$$
\varphi(z)=0, \quad \psi(z)=z^{\lambda-1}, \qquad H(z,\bar{z})=I_\lambda(kr)\sin\lambda\vartheta \qquad (3.7)
$$

where I_λ is a Bessel function of the first kind with an imaginary argument, and $\lambda = \pi/\alpha$.

For values of angle α, which satisfy the condition $\frac{1}{2}\pi<\alpha<\pi$, solution (3.7) for $r\to 0$ gives bounded displacements and couple stresses, while ordinary stresses will be unbounded of the order lower than unity; more exactly, the stresses will have the order $O(r^{\lambda-2})$, where $0<2-\lambda<1$. We shall obtain the same picture if in (3.7) for $\pi<\alpha<2\pi$ the value $\lambda=2\pi/\alpha$ is taken.

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